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THE LAGRANGIAN APPROACH OF ADVECTIVE TERM TREATMENT AND ITS APPLICATION TO THE SOLUTION OF NAVIER–STOKES EQUATIONS

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SUMMARY

A one-dimensional transport test applied to some conventional advective Eulerian schemes shows that linear stability analyses do not guarantee the actual performances of these schemes. When adopting the Lagrangian approach, the main problem raised in the numerical treatment of advective terms is a problem of interpolation or restitution of the transported function shape from discrete data. Several interpolation methods are tested. Some of them give excellent results and these methods are then extended to multi-dimensional cases.

The Lagrangian formulation of the advection term permits an easy solution to the Navier–Stokes equations in primitive variables V, p, by a finite difference scheme, explicit in advection and implicit in diffusion.

As an illustration steady state laminar flow behind a sudden enlargement is analysed using an upwind differencing scheme and a Lagrangian scheme. The importance of the choice of the advective scheme in computer programs for industrial application is clearly apparent in this example.

KEY WORDS Lagrangian Advective Schemes Numerical Diffusion Navier–Stokes

1. INTRODUCTION

In fluid mechanics, it is well known that the treatment of advection hyperbolic terms is particularly difficult.

A very simple differential equation such as:

$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}=0 \tag{1}$$

where $F(x, t)$ is an unknown scalar function and $u(x, t)$ a given scalar function cannot, for the general case, be satisfactorily solved at the present time. This is amply demonstrated by the abundance of available literature. We will define two approaches to the problem:

—the Eulerian approach, which is a conventional method to solve this type of equations with partial derivatives. Assuming that the function are indefinitely derivable in x and t, we try to find a formulation utilizing finite differences and fitting with a Taylor's series up to the highest possible order,

—the Lagrangian approach, which is a physical rather than numerical method and suited to solving a transport problem. Knowing function $F(x, t_1)$ at time t_1 for any x, we determine the values $F(x, t_2)$ at time t_2 assuming that the function F remains constant over the

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characteristic curves of plane (x, t) which are the solutions of the differential equation:

$$\frac{dx}{dt} = u(x, t)$$

From the available bibliography, it appears that the method studied most often is the Eulerian approach. This is quite clear since time step limitations are not as stringent when utilizing implicit methods particularly when steady state conditions are sought. However, when limiting the study to industrial incompressible flows, an implicit advective scheme does not appear to be essential. In actual fact, grids are seldom refined in the direction normal to streamlines. Consequently, the use of an explicit scheme in advection may result in a quite acceptable computation time, provided it is implicit in diffusion. Therefore, there is no reason to discard the essentially explicit Lagrangian schemes.

Moreover, if the performance of the scheme is ignored and quality of the results is considered in isolation, the literature remains rather prudent on the effect of the choice of the advective schemes on the solution of problems where advection is not the only physical mechanism involved. However, it obviously is a vital question when computing industrial incompressible flows where high accuracy is required. It then becomes interesting to go beyond pure advection tests and to compare the solutions of the Navier–Stokes equations related to a single steady flow as obtained through different advective schemes.

2. REVIEW OF SOME EULERIAN METHODS

The methods studied here are well established in the literature. They have been widely investigated with regard to consistency and linear stability. These will not, therefore, be discussed in detail. In the present text the different methods are compared via a one-dimensional test as defined below.

2.1. *Definition of the one-dimensional test*

We revert to the problem presented in section 1 with the x-axis discretized with a constant step δx and the time-axis discretized with a step δt. Thus, $x = x_0 + i\delta x$, $t = t_0 + n\delta t$ and equation (1) may be conveniently expressed in the reduced form:

$$\frac{\partial F}{\partial n} + c\frac{\partial F}{\partial i} = 0 \tag{2}$$

where c is the dimensionless Courant number

$$c = \frac{u\delta t}{\delta x}$$

A strict one-dimensional test allows an in-depth visualization of the essential properties of schemes investigated:

—numerical diffusion

—phase error

—stability

Taking the example proposed by Holly *et al.*:[1] the function $F(i, 0) = 10e^{-0 \cdot 4i^2}$ is transported over 24 time steps (respectively 38, 48, 96) with $c = 1$ (respectively 0·75, 0·5, 0·25).

2.2. Leith's method

It is well known that the 'natural' (centred) discretization of equation (2)

$$F_i^{n+1}-F_i^n+\frac{c}{2}(F_{i+1}^n-F_{i-1}^n)=0 \tag{3}$$

is unstable and that the upwind differencing scheme or FTUS1 (Forward Time Upwind Space of first order)

$$F_i^{n+1}-F_i^n+c(F_i^n-F_{i-1}^n)=0 \tag{4}$$

in spite of having excellent stability and phase error properties (and, from a more practical viewpoint, cost and simplicity advantages) exhibits numerical diffusion, Figure 1.

A compromise can therefore, be sought between formulations (3) and (4); this can be accompanied, for instance, by introducing a variable shift parameter, such as proposed by Hirt *et al.*:[2]

$$F_i^{n+1}-F_i^n-c\left[\lambda(F_i^n-F_{i-1}^n)+\frac{1-\lambda}{2}(F_{i+1}^n-F_{i-1}^n)\right]=0 \tag{5}$$

The authors have endeavoured to minimize this shift in order to ensure the linear stability as defined by von Neumann. After rather tedious computations, it can be demonstrated that the scheme is stable if $\lambda > c$ and unstable if $\lambda < c$ (taking the usual restrictions $c \leqslant 1$). The minimum shift therefore corresponds to $\lambda = c$ and we obtain Leith's formulation:

$$F_i^{n+1}-F_i^n+\frac{c}{2}(F_{i+1}^n-F_{i-1}^n)-\frac{c^2}{2}(F_{i+1}^n+F_{i-1}^n-2F_i^n)=0 \tag{6}$$

The scheme has interesting theoretical properties and is often shown in the literature under various forms, for instance by Roache,[3] Dukowicz and Ramshaw,[4] Godunov and MacCormack as quoted by Sod.[5] However, for the present test case the results were rather poor, Figure 2.

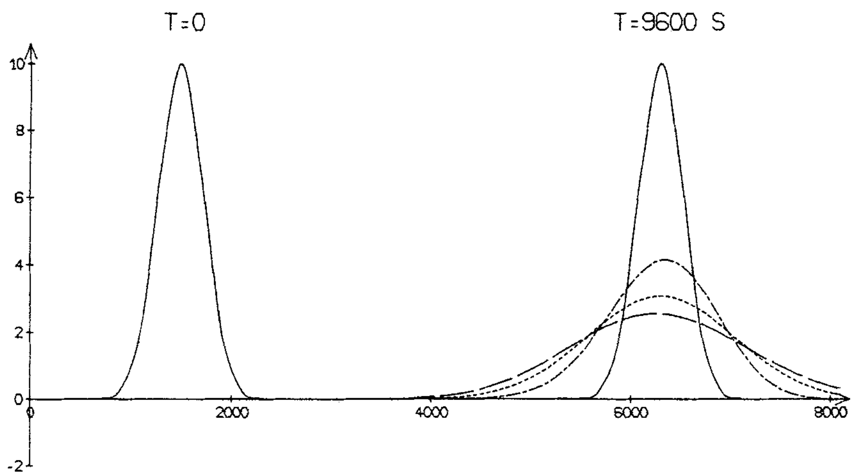

Figure 1. Upwind differencing scheme (FTUS1)

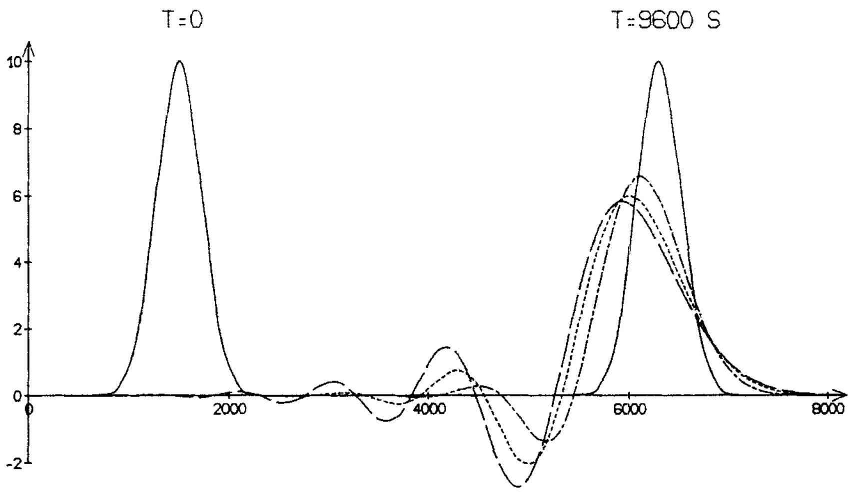

Figure 2. Leith scheme

2.3. Euler's implicit method (2)

This is a centred scheme, stabilized by a method of Milne's type and rendered semi-implicit:

$$F_i^{n+1} - F_i^n + \frac{c}{4}(F_{i+1}^{n+1} + F_{i+1}^n - F_{i-1}^{n+1} - F_{i-1}^n) = 0 \tag{7}$$

The theoretical properties of this scheme are excellent:

—truncation error of $O(\delta x^2, \delta t^2)$

—amplification factor of Neumann's analysis $|V^{n+1}/V^n|$ identically equal to 1, thus suggesting an unconditional stability scheme without numerical diffusion.

The results from the test case, Figure 3, gave poorer results than those obtained with

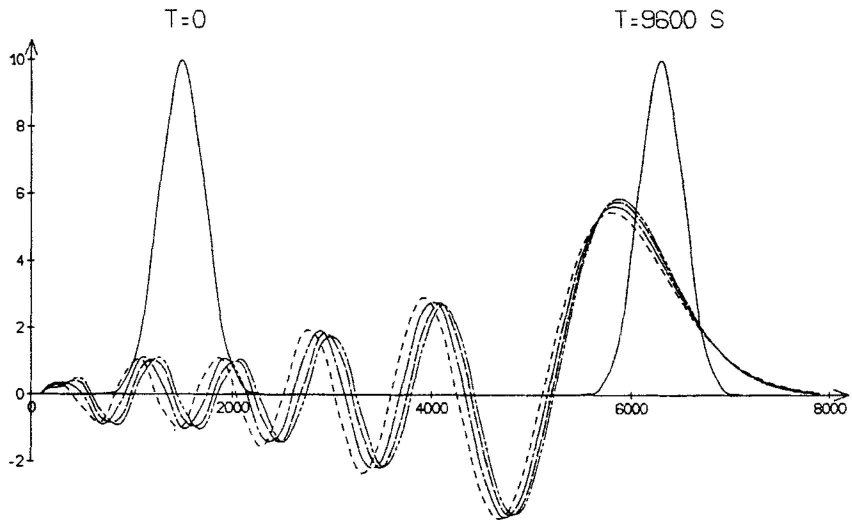

Figure 3. Euler's modified method

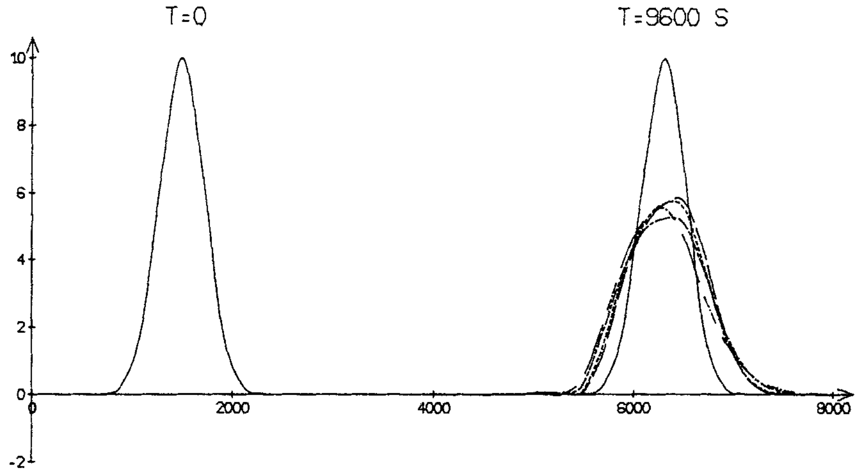

Figure 4. Petula

Leith's method. However, it must be noted that the solution is only slightly dependent on the Courant number. Owing to this interesting property, the scheme constitutes the starting point of the advective scheme of the PETULA program[6,7] which was finalized in 1968. Several improvements have been made to scheme (7) and in particular:

—discretization of fourth-order spatial derivatives

—correction of non-realistic values

—reconstruction of the integral value of the solution at time $n+1$ such as:

$$\int F^{n+1}\,dx = \int F^{n}\,dx$$

This rather dated method yields results which are particularly useful for present purposes, Figure 4. It must, nevertheless, be stressed that due to the tests involved, this method is non-linear, expensive and difficult to program effectively.

2.4. Other implicit methods

Another way of stabilizing the centred scheme, (3), is to write it in a fully implicit form:

$$F_i^{n+1} - F_i^n + \frac{c}{2}(F_{i+1}^{n+1} - F_{i-1}^{n+1}) = 0 \qquad (8)$$

According to Roache,[3] the amplification factor of this scheme is:

$$|G|^2 = \frac{1}{1 + c^2 \sin^2 \theta}$$

This exhibits unconditional stability but a numerical diffusion which increases with increasing c, such as shown in Figure 5. Consequently, transient flows could not be effectively studied using such a scheme.

The FTUS1 can also be written in the implicit form:

$$F_i^{n+1} - F_i^n + c(F_i^{n+1} - F_{i-1}^{n+1}) = 0 \qquad (9)$$

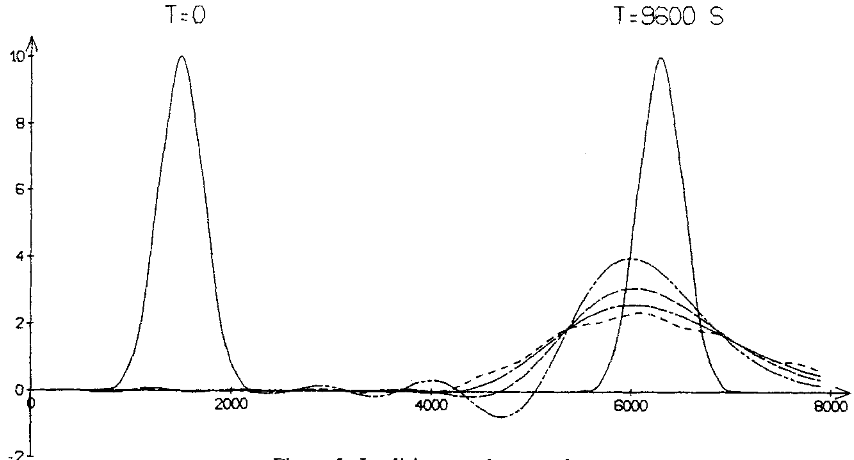

Figure 5. Implicit centred space scheme

This certainly is the simplest, and most currently used, way of obtaining a steady state solution. According to our experience, scheme (9) has a more reliable behaviour than scheme (8).

3. THE LAGRANGIAN APPROACH

The Lagrangian approach related to the transport problem has been investigated, amongst other authors, by Roache[3] (p. 75). Taking F as a given property of the fluid and if we write (1) in the form:

$$\frac{DE}{Dt}=0 \tag{10}$$

indicates that the value of function F specific to a fluid particle remains constant in time. We will generalize this concept in section 4 but at this stage, we revert to the conditions and notations of subsection 2.1 where function F is known at the points x_i of a discretized space (x, t). The values of F_i^{n+1} are required for a constant velocity field u. Referring to Figure 6, the fluid particle which is located at point x_i at time t_{n+1} was located at point x^* at time t_n. The overall transport equation(s) can therefore be split into two separate problems,

(i) localization of x^* or, in other words, determination of the characteristic curve crossing (x_i, t_{n+1}). Under the elementary conditions of the test case we have,

$$x^* = x_i - u\delta t$$

or, in reduced co-ordinates,

$$i^* = i - c$$

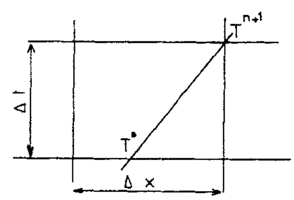

Figure 6. Trajectory of the particle

and

(ii) interpolation of a value F^* at point x^*, according to neighbouring nodal values, at time t_n.

Equation (10), therefore, becomes,

$$F_i^{n+1} - F^* = 0 \tag{11}$$

in which the computation of F^* is the critical feature of this method.

3.1. *Problems related to interpolation*

First, let us note that if c is an integer, we immediately have,

$$F_i^{n+1} = F^* = F_{i-c}^n$$

But in the general case, an interpolation function must be selected. In order to stress the importance of this choice, let us first consider linear interpolation which obviously leads to the explicit FTUS1 scheme (4). Consider the transport of the trapezium of Figure 7, as defined by $F_i = 0$ for any i except when $i = 4, 5, 6$, for which $F_i = 1$ over three

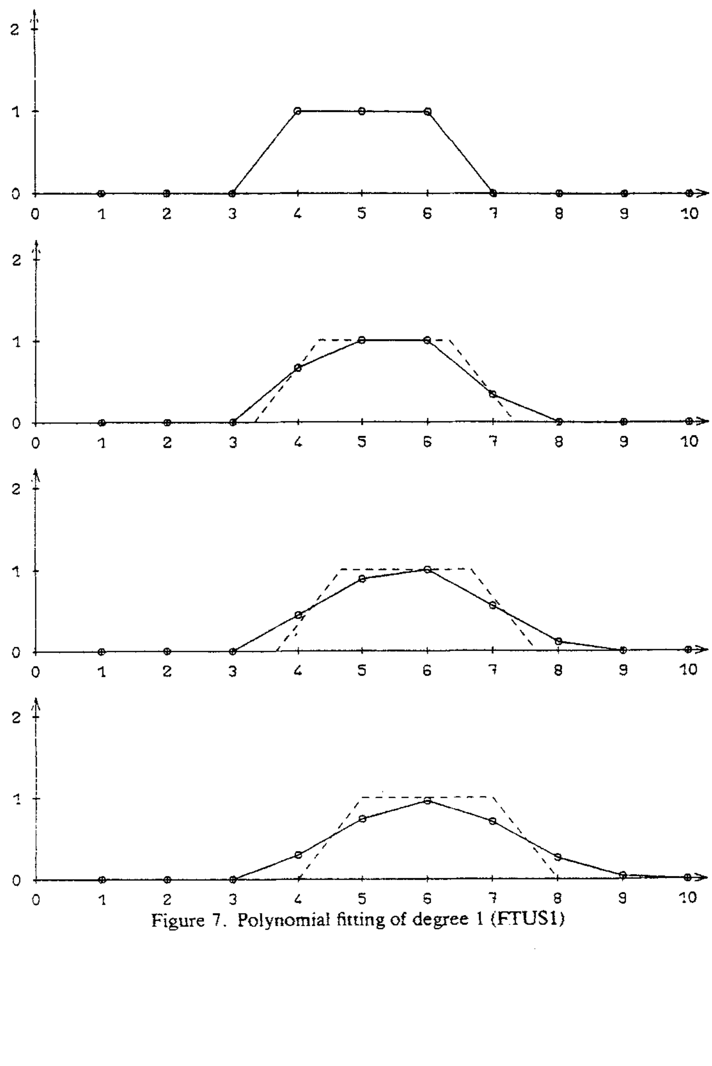

Figure 7. Polynomial fitting of degree 1 (FTUS1)

time steps, with $c = 1/3$. The first interpolation gives the values circled on Figure 7(b) from which the linear interpolation is, of course, not capable of regenerating a trapezium. It can only reproduce the polygon shown by the solid lines which are already markedly affected by numerical diffusion. This polygon is used as the basis of the second interpolation which will produce another polygon of a still more flattened type and so on. When three time steps have been completed, the trapezium is hardly recognizable. This example clearly shows the basic difficulty raised by the numerical treatment of advective terms: if the selected interpolation function could be adapted to the transported function, it would be capable of reconstituting the shape of this function from the discrete data issued from the first interpolation. For instance, the linear interpolation transports a straight line in an accurate way and without any restrictions depending on the Courant number. Similarly, if a perfect result were desired with our test case of subsection 2.1, we would simply select the interpolation function with three parameters,

$$\alpha \exp[\beta(i-\gamma)^2]$$

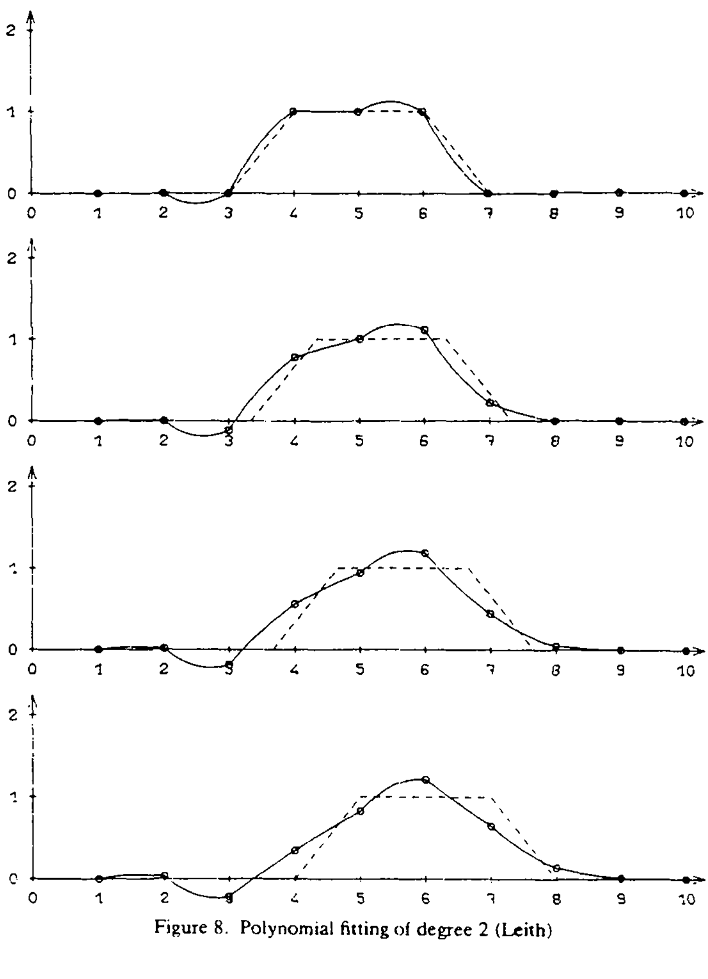

Figure 8. Polynomial fitting of degree 2 (Leith)

However, in practice, a single interpolation function is used for all the computations and hence the transported function itself has to suit the interpolation function, with associated varying magnitude distortions.

3.2. *Polynomial interpolation*

A local interpolation method is therefore sought with associated restrictions on the Courant number (generally: $c \leqslant 1$). Polynomial interpolations of second and third order appear to be of particular use. The second-order centred interpolation on points $i-1$, i, $i+1$, gives Leith's formulation (6). The trapezium transported by this method is indicated in Figure 8. The third-order interpolation is not centred and applies to points $i-2$, $i-1$, i, $i+1$. The trapezium transport with this scheme is shown in Figure 9 and the results of the test case, subsection 2.1, are shown in Figure 10. It is apparent the third-order interpolation gives satisfactory results which are much better than those obtained by Leith's method. In

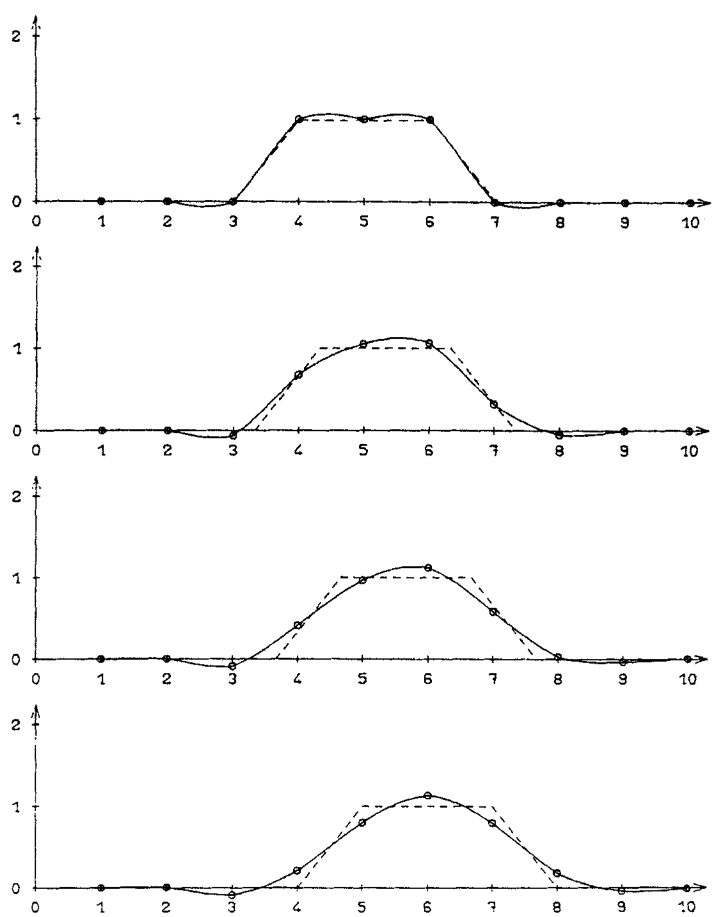

Figure 9. Polynomial fitting of degree 3 (FTUS3)

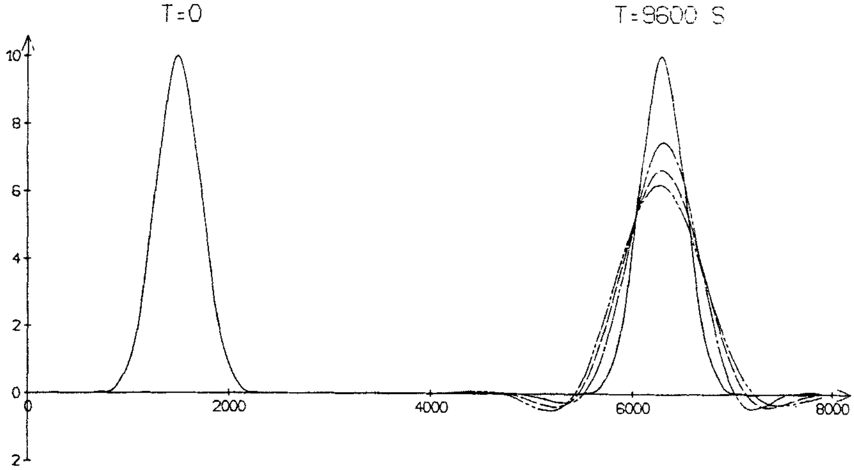

Figure 10. Polynomial fitting of degree 3 (FTUS3)

this paper, this scheme will now be referred to as FTUS3 (Forward Time Upwind Space of order 3). The trapezium transport tests have been conducted with polynomials of higher order with no significant improvement in the result.

Scheme FTUS3 can be expressed as follows:

$$F_i^{n+1}-F_i^n+c\left(\frac{F_{i-2}^n}{6}-F_{i-1}^n+\frac{F_i^n}{2}+\frac{F_{i+1}^n}{3}\right)$$
$$-\frac{c^2}{2}(F_{i-1}^n-2F_i^n+F_{i+1}^n)-\frac{c^2}{6}(F_{i-2}^n-F_{i+1}^n+3(F_i^n-F_{i-1}^n))=0 \quad (12)$$

From von Neumann's linear analysis, this scheme is conditionally stable for $c \leqslant 1$ but this restriction may certainly be relaxed up to $c \leqslant 2$. With a constant mesh, the truncation error is $O(\delta t, \delta x^3)$. When comparing Figures 1 and 10, it can be seen that scheme FTUS3 largely retains the good properties of FTUS1, such as stability and, more significantly, phase error, while the numerical diffusion is reduced considerably. Scheme FTUS3, similar to FTUS1 is linear and may have an implicit formulation if only steady state problems are considered.

3.3. *Hermite interpolation*

Another conventional interpolation method consists of determining the derivatives $(\partial F/\partial i)_i$ and $(\partial F/\partial i)_{i-1}$ and then interpolate with a third-order polynomial and checking the values of function F and of its derivative at points i and $i-1$. A brief review of two methods based on this procedure will be included.

3.3.1. *Derivative transport.* The explicit scheme of very high accuracy is proposed by Holly *et al.*[1] A transport equation for the derived function is required. Under the elementary conditions of the test case, the derivative of equation (2) is merely,

$$\frac{\partial}{\partial n}\left(\frac{\partial F}{\partial i}\right)+c\frac{\partial}{\partial i}\left(\frac{\partial F}{\partial i}\right)=0$$

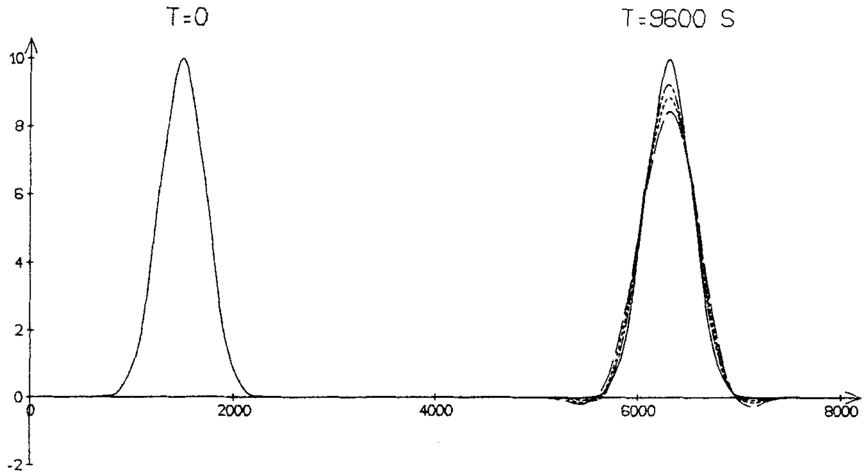

Figure 11. Transport of the derivatives

At time n, F_i, F_{i-1}, $(\partial F/\partial i)_i$, $(\partial F/\partial i)_{i-1}$ are assumed known and a Hermite polynomial is defined which allows F^* and $(\partial F/\partial i)^*$ to be determined. Consequently, the values of the function and of its derivative at time $n+1$ are simply,

$$F_i^{n+1} = F^*$$

and

$$(\partial F/\partial i)_i^{n+1} = (\partial F/\partial i)^*$$

The results of the test case are excellent, Figure 11. According to the authors, they can be further improved by transporting, in the same way, the second-order derivative. There is, theortically, no reason why this method cannot be used in the solution of multi-dimensional Navier–Stokes equations. However, the momentum equations must be derived to the second order for the plane problems and to the third order for the three-dimensional problems. This therefore induces a significant computation complexity and associated prohibitive cost.

3.3.2. *Akima's*[8] *non-linear formula.* In Akima's method, derivative $(\partial F/\partial i)_j^n$ is computed in terms of five values from F_{i-2}^n to F_{i+2}^n. Writing

$$m_j = F_{j+1}^n - F_j^n, \qquad j = i-2, i+1$$

then

$$(\partial F/\partial i)_i^n = (|m_4 - m_3|\, m_2 + |m_2 - m_1|\, m_3)/(|m_4 - m_3| + |m_2 - m_1|)$$

If the denominator is zero

$$(\partial F/\partial i)_i^n = (m_2 + m_3)/2$$

and the derivative $(\partial F/\partial i)_{i-1}$ can also be computed in a similar manner, using the values F_{i-3}^n to F_{i+1}^n. The advective scheme requires six points F_{i-3}^n to F_{i+2}^n as a total. The results for the test case, Figure 12, are very good and this scheme appears to be basically interesting in spite of two obvious drawbacks—the non-linearities and even the possible discontinuities of derivatives due to the presence of absolute values and the large number of points required.

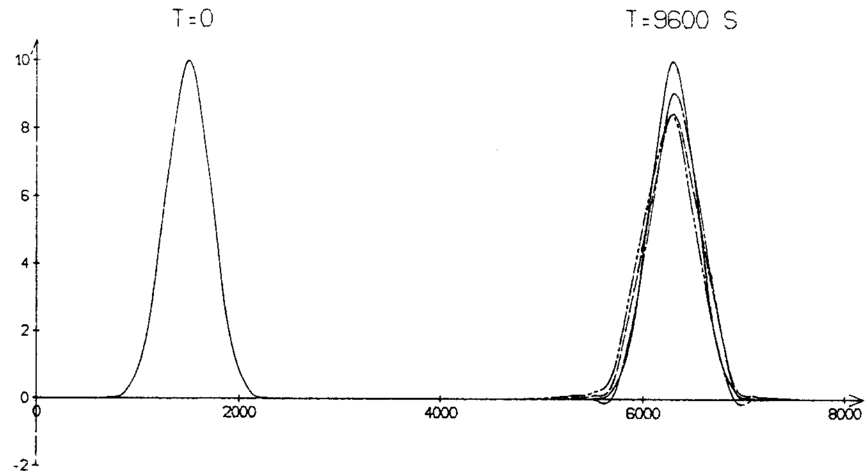

Figure 12. Akima interpolation method

4. TWO-DIMENSIONAL TRANSPORT

Two-dimensional transport for Lagrangian schemes only are considered. As in the previous section, a local interpolation method is required.

4.1. *Two-dimensional test case*

On a constant-mesh grid (x_i, y_j) define the square,

$$F_{i,j} = 1 \quad \text{if} \quad 4 \leq i \leq 6,\ 4 \leq j \leq 6$$

and

$$= 0 \quad \text{in the other cases.}$$

which is transported over 20 time steps with $c_x = 0{\cdot}4$ and $c_y = 0{\cdot}3$. The result of this transport is shown by a cross-section in the i and a cross-section in the j directions.

4.2. *'Uncoupled' directions*

This consists of the successive application of a one-dimensional method in the x and, separately, in the y directions to obtain the values F_x^* and F_y^* and

$$F^* = F_x^* + F_y^* - F_{i,j}^n \tag{13}$$

It can be stated that most of the advective schemes, in particular the Eulerian schemes, are based on this principle. However, this procedure is not logical and may lead to aberrations in some limiting cases. We readily see that (13) features a linear extrapolation via the plane crossing points F_x^*, F_y^* and $F_{i,j}^n$. The necessity of determining the intermediate values F_x^*, F_y^* by means of sophisticated schemes, can then be questioned if the computation is completed by a rough extrapolation. From a strictly logical viewpoint, the only one-dimensional scheme adapted to formulation (13) is that which is itself based on linear interpolation, namely the FTUS1 scheme, Figure 13.

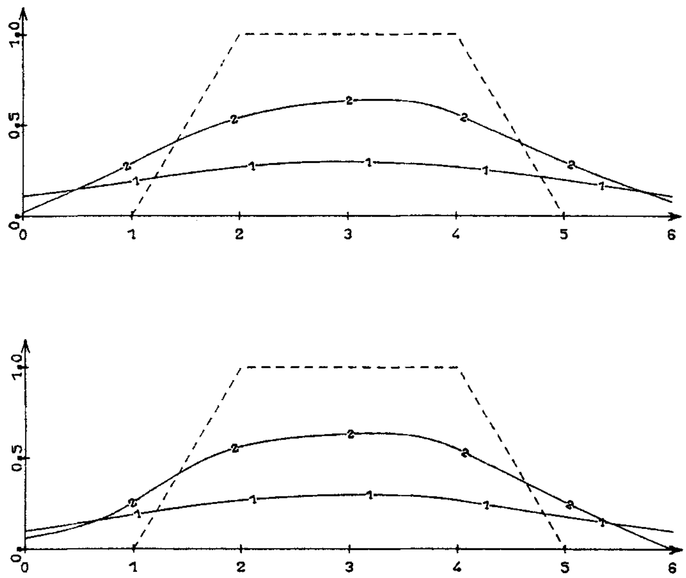

Figure 13. Two-dimensional test problem: 1—FTUS1, 2—Petula

The PETULA scheme (subsection 2.3) falls into the category of 'uncoupled' schemes. It gives acceptable results, Figure 13, owing to the corrective-compensating algorithm. However, the results are rather bad when FTUS3 and Akima's schemes, Figure 14, are used.

We will explain, below, how to improve the accuracy of (13) while increasing, slightly, the cost and complexity of solution.

4.3. Two-dimensional interpolation

The logical expansion of the FTUS3 scheme to the two-dimensional case consists of using surface interpolated over points $[i-2, i+1]\times[j-2, j+1]$ if the velocity components are positive. This surface is described by the polynomial,

$$F(x, y)=\sum_{m=0}^{3}\sum_{n=0}^{3} a_{m,n}x^m y^n \tag{14}$$

In fact, instead of carrying out directly the computation of the 16 coefficients $a_{m,n}$, it is more advisable to make five one-dimensional interpolations such as indicated in Figure 15.

The Akima scheme can, in a similar manner be converted to the two-dimensional case but through seven one-dimensional interpolations as this scheme uses six points.

The Holly *et al.*[1] method leads to the simultaneous treatment of four equations of transport in F, $\partial F/\partial x$, $\partial F/\partial y$, $\partial^2 F/\partial x\,\partial y$ and to the effective computation of the 16 coefficients of a polynomial of type (14). Results are still excellent but the cost, for the simple problem of transport in a constant velocity field, is 10 times higher than the other two schemes. Figure 16 shows the results of the test case for these three schemes, all of which appear acceptable.

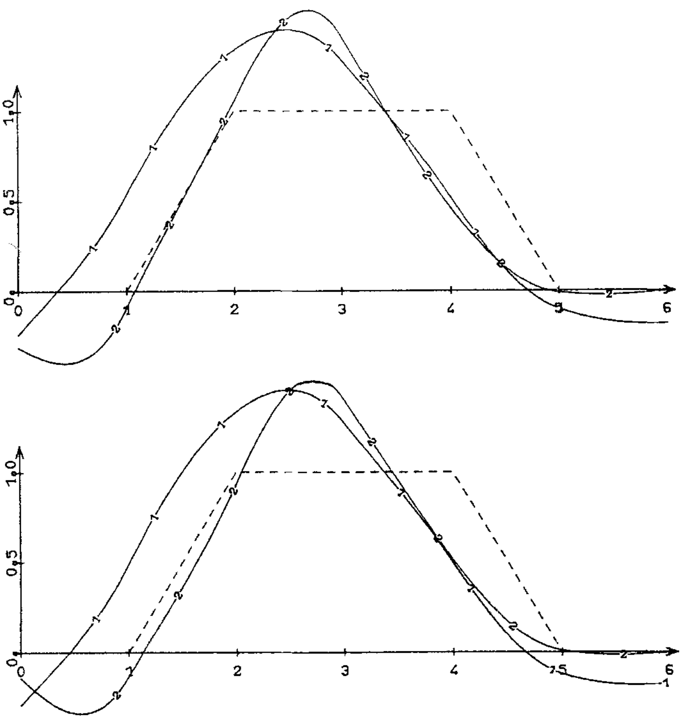

Figure 14. Two-dimensional test problem: 1—FTUS3 uncoupled, 2—Akima uncoupled

4.4. *Simplified interpolation*

In this section, we propose an interpolation method which is intermediate, regarding cost and complexity, between the two previous methods. A positive velocity field ($u>0$, $v>0$) is considered and the interpolation takes place in the area $[i-1, i]\times[j-1, j]$. Applying the uncoupled direction method regarding i and y, as in subsection 4.2, the following values are

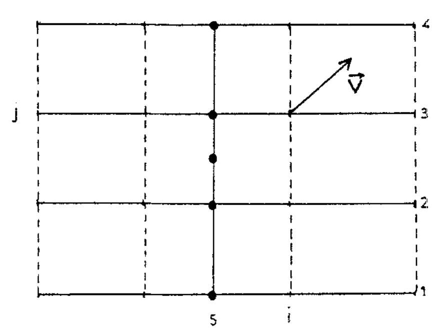

Figure 15. Two-dimensional interpolation for FTUS3

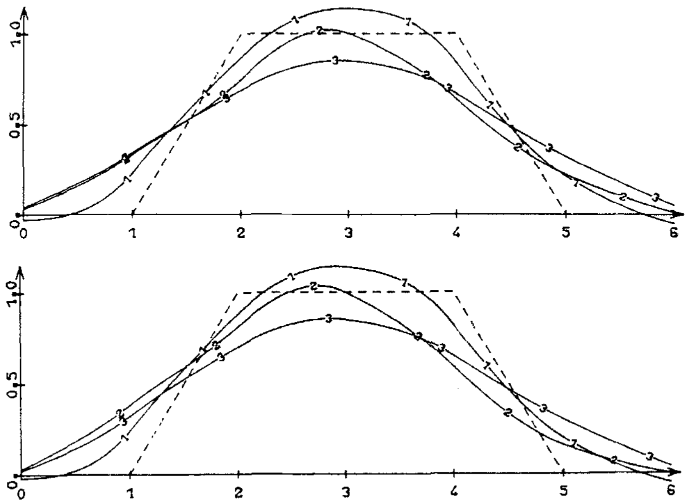

Figure 16. Two-dimensional test problem: 1—Transport of the derivatives, 3—Akima, 3—FTUS3

obtained

$$(\partial F/\partial x)_{i-1,j} \qquad (\partial F/\partial x)_{i,j}$$
$$(\partial F/\partial y)_{i,j-1} \qquad (\partial F/\partial y)_{i,j}$$

To these four values are added the values of the function at four corners of the interpolation area. These eight values are sufficient to determine an eight-coefficient polynomial which, considering the symmetry and maximum degree, is,

$$F(x, y) = a_1 + a_2x + a_3y + a_4xy + a_5x^2 + a_6y^2 + a_7x^3 + a_8y^3$$

The results, shown in Figure 17, of this simplified method differ only slightly from those of Figure 16. The cost is hardly higher than that of the 'uncoupled' method (13) for much more exacting solutions.

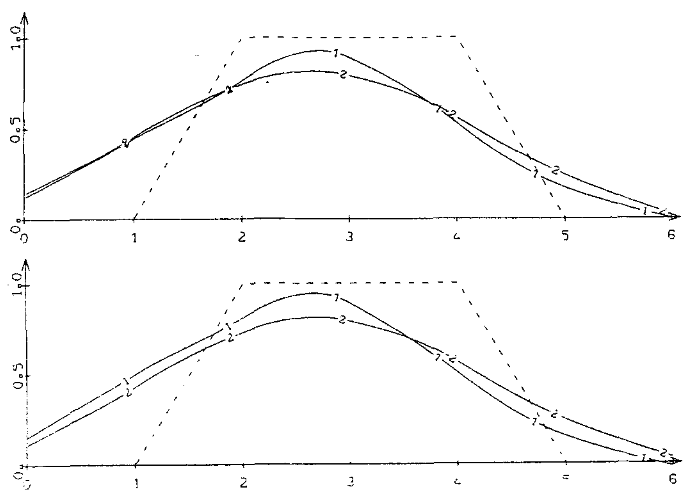

Figure 17. Two-dimensional test problem: 1—Akima simplified, 2—FTUS3 simplified

5. SOLUTION OF THE NAVIER–STOKES EQUATIONS

5.1. Generalization of the Lagrangian approach

It is not difficult to apply a full Lagrangian approach to the advection–diffusion problem in a multi-dimensional space. Consider a fluid in motion and to each fluid particle M, assign a scalar value $F(M, t)$ which varies in time according to the law

$$\frac{DF}{Dt}(M, t) = \phi(M, t) \tag{15}$$

Assuming that at time t_1, particle M is located at M_1 and at time t_2, at M_2 and integrating equation (15) between times t_1 and t_2 we obtain

$$F(M_2, t_2) - F(M_1, t_1) = \oint_{t_1}^{t_2} \phi(M, t)\,dt \tag{16}$$

Formulation (16) applies directly to the discrete problem in which t_1 corresponds to time n, t_2 to time $n+1$, M_2 is the position of node k in the spatial mesh system, M_1 is an M^* value which does not generally correspond to a node and $F(M_1, t_1)$ is the interpolated value F^*, from the neighbouring nodal values, at time n.

$$F_k^{n+1} - F^* = \oint_{t_n}^{t_{n+1}} \phi(M, t)\,dt \tag{17}$$

Equation (17) is still exact but the right-hand term requires two simplifications in order to obtain a workable scheme. First, the curvilinear integral is replaced by a simple integral

$$F_k^{n+1} - F^* = \int_{t_n}^{t_{n+1}} \phi(M_k, t)\,dt$$

This simple integral can be estimated in different ways, including the three conventional methods viz. $\phi^n \delta t$, $\frac{1}{2}(\phi^n + \phi^{n+1})\delta t$, $\phi^{n+1}\delta t$. The third method is adopted and equation (15) can be written in the discrete form

$$F_k^{n+1} - F^* = \phi_k^{n+1}\delta t \tag{18}$$

5.2. Navier–Stokes equations

Consider the two, or three, dimensional system for the equations of the laminar motion of Newtonian fluids.

$$\nabla V = 0 \quad \text{—continuity} \tag{19}$$

and

$$\frac{DV}{Dt} = -\frac{1}{p}\nabla p + \nu \nabla^2 V \quad \text{—momentum} \tag{20}$$

In accordance with subsection 5.1, this system can be written as

$$\nabla V^{n+1} = 0 \tag{21}$$

$$(V_k^{n+1} - V^*)/\delta t = -\frac{1}{\rho}\nabla p^{n+1} + \nu \nabla^2 V^{n+1} \tag{22}$$

in which the term V^* corresponds to $V_k^n - \delta t(V\nabla V)_k^n$. Equation (22) already constitutes a discrete approximation of (20) but the operators ∇ are not yet discretized.

Taking the divergence of both sides of (22) and utilizing equation (21) we have

$$\nabla^2 p^{n+1} = \rho \nabla V^* / \delta t \tag{23}$$

It should be noted at this juncture that the field V^* is not conservative. The question is now posed: Under what conditions is system (22), (23) equivalent to system (21), (22)? Assuming that p^{n+1} and V^{n+1} are solutions of (22), (23), taking the divergence of (22), but this time incorporating (23) we have

$$\nabla V^{n+1} = \nu \nabla^2 (\nabla V^{n+1}) \tag{24}$$

When imposing the boundary condition $\nabla V^{n+1} = 0$ at the boundaries of the computation field, equation (24) results in the trivial solution (21) only.

5.3. *Resolution algorithm*

Assuming that V^n and p^n are known, V^{n+1} and p^{n+1} can be calculated in three steps

5.3.1. *Step 1—Advection.* This step involves two separate problems viz. localization of point M^* and interpolation of value V^*. To resolve the first problem normally requires calculation of the equation:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = V(M, t)$$

starting from point M_k, and between times t and $t - \delta t$. However, V^{n+1} is not known at this stage. Therefore, we assume that V does not vary over this time step and simply solve:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = V^n(M) \tag{25}$$

The components of V^n can be expressed in analytical form near point M_k and equation (25) can then be solved by a single step of Runge–Kutta's fourth-order algorithm. This method of localization will be called 'the method of the curvilinear characteristic'.

Many Eulerian methods utilize the assumption of the rectilinear characteristic

$$\frac{\mathrm{d}M}{\mathrm{d}t} = V_k^n$$

The second problem is interpolation which has already been discussed in detail in section 4.

5.3.2. *Step 2—Pressure solution.* This step consists of solving equation (23) with boundary conditions usually of the Neumann type. It is therefore prudent to solve the discrete system of equations by means of a direct method.

5.3.3. *Diffusion.* The pressures being known, the elliptic equations (22) are then solved to complete the solution. In this case, the matrices of the systems are strictly diagonally dominant and an over-relaxation method is advisable.

5.4. *Boundary conditions*

A tangential condition and a normal condition on each boundary of the computation field are required to solve the equation system (19), (20). The three elliptic systems (22), (23), in the plane case, require a boundary condition on these boundaries, one of these three conditions must be tangential, the other normal and the third one simply inferred from the other two.

Consider, for instance, a solid horizontal wall, aligned in the x-direction,

$u = 0$ is the tangential condition
$v = 0$ is the normal condition

An expression for the normal pressure gradient $(\partial p/\partial y)$ is derived from the second momentum equation (22), or, in steady state

$$\partial p/\partial y = \rho\nu\, \partial^2 v/\partial y^2$$

5.5. *Advantages in the Lagrangian formulation*

We have seen that V^* is the discretization of term $V_k^n - (V.\nabla V)_k^n \delta t$. The term ∇V^* of Poisson's equation (23) therefore contains implicitly the stabilizing term ∇V_k^n which is found in all the methods using primitive variables V, p. Consequently, for an 'uncoupled' discretization of the advective terms, subsection 4.2, and incorporating the assumption of rectilinear characteristics, subsection 5.3.1, the formulation in V^* only is a convenient way of writing the right-hand side of Poisson's equation (23). However, when associated with the multidimensional interpolation and to the method of the curvilinear characteristic, it effectively improves the accuracy of the advective scheme.

6. COMPUTATION OF A STEADY-STATE TWO-DIMENSIONAL FLOW

It is of course of interest to illustrate the algorithm described under section 5 by example. However, the prime current intention is to stress the importance of the selected advective scheme in a practical computation of Newtonian viscous flow. This matter is of importance since, at the present time, many existing programs incorporate upwind differencing schemes which lead, particularly in transient flow conditions, to unacceptable numerical diffusion. This may contribute to the fact that fast transient flows of viscous fluids are not studied too often. However, in steady state, if the streamlines are parallel to the mesh lines, there is no diffusion when the FTUS1 scheme is used. Unfortunately, the streamlines cannot always be parallel to the mesh lines and in such a case, the numerical diffusion of FTUS1 may be significant, even under steady state conditions.

6.1. *Definition of computation*

The particular example studied is the flow behind a sudden enlargement, in the computation field shown in Figure 18. Distances are normalized with respect to the height, L, of the step. The grid has a 10×20 constant-size mesh with $\delta x = 1$, $\delta y = 0{\cdot}2$. The velocity profile at the inlet is constant $u = V_0$, $v = 0$. Several computations are conducted corresponding to increasing values of Reynolds' number $\mathrm{Re} = V_0 L/\nu$. For each value of Re, the streamlines are compared for the solutions obtained using the FTUS1 scheme and the simplified FTUS3 scheme of subsection 4.4.

6.2. *Results*

Three computation series were conducted, for Re = 1000, 10,000 and 10^6 which relates to turbulent flow. These computations for laminar flow although unrealistic serve to demonstrate the trend with increasing Reynolds number.

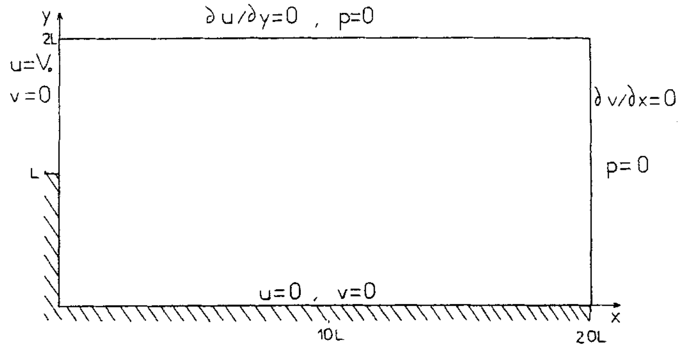

Figure 18. Flow down a step: geometry and boundary conditions

The six solutions are shown in Figures 19–24 in the form of iso-values of stream functions. The dimensionless length of main flow re-attachment is presented in Table I:

Table I. Re-attachment length values

Scheme \ Re	1000	10,000	1,000,000
FTUS1	6	6·5	6·7
FTUS3	7·5	9	12

It can be noted that the difference between the two solutions is more apparent as the Reynolds' number increases. Furthermore, it should be noted that the solution obtained with the FTUS1 scheme results in small variations for a Reynolds' number equal to, or higher than, 1000, illustrating that numerical diffusion is prevailing over the molecular diffusion.

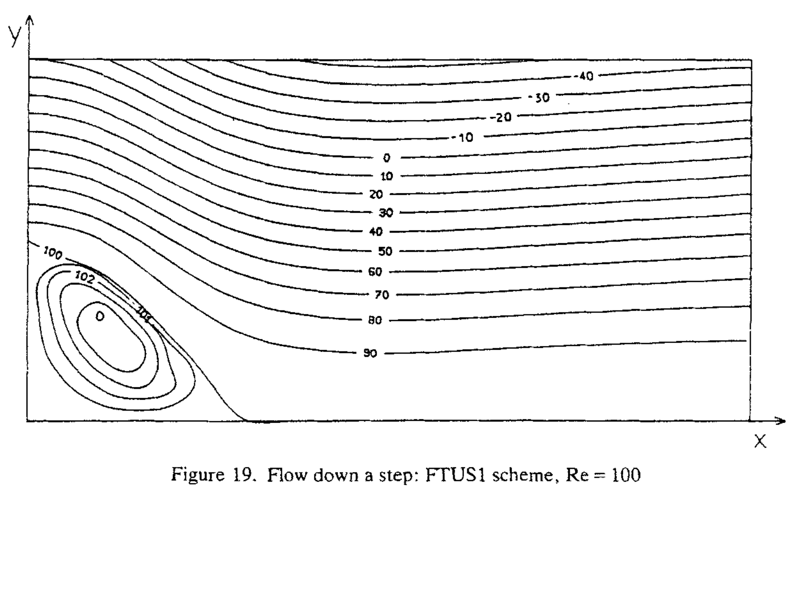

Figure 19. Flow down a step: FTUS1 scheme, Re = 100

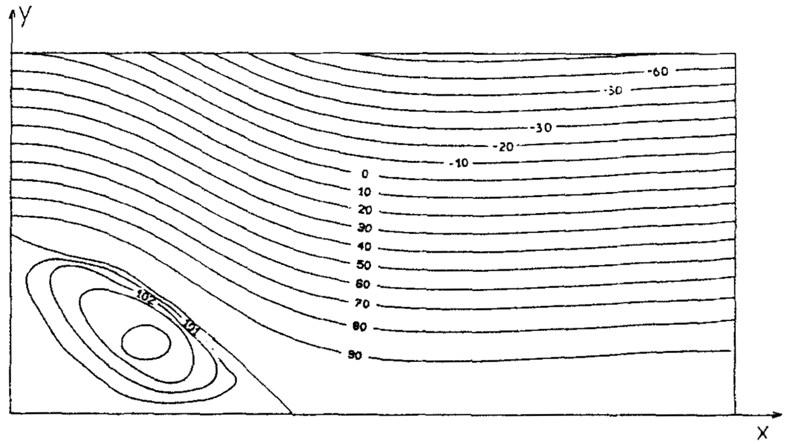

Figure 20. Flow down a step: FTUS3 scheme, Re = 100

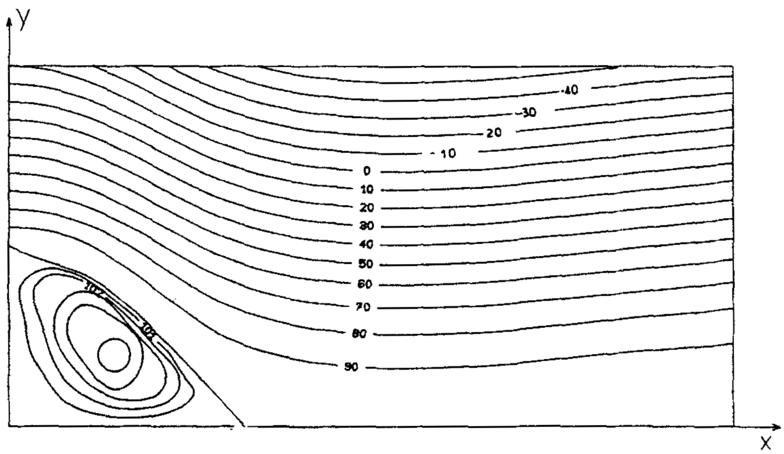

Figure 21. Flow down a step: FTUS1 scheme, Re = 10,000

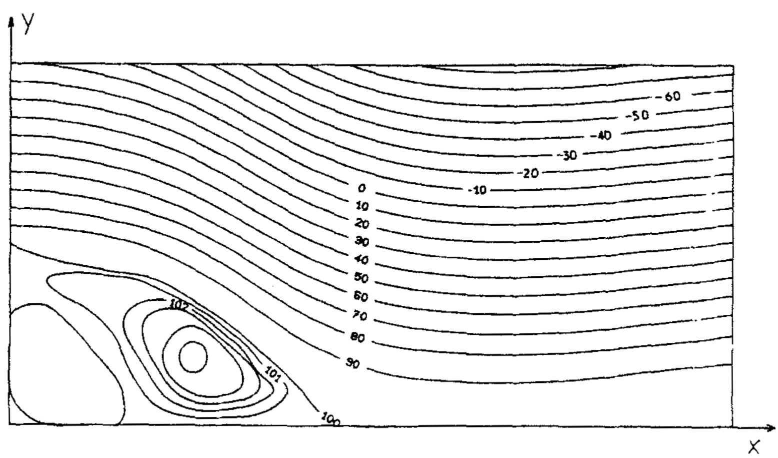

Figure 22. Flow down a step: FTUS3 scheme, Re = 10,000

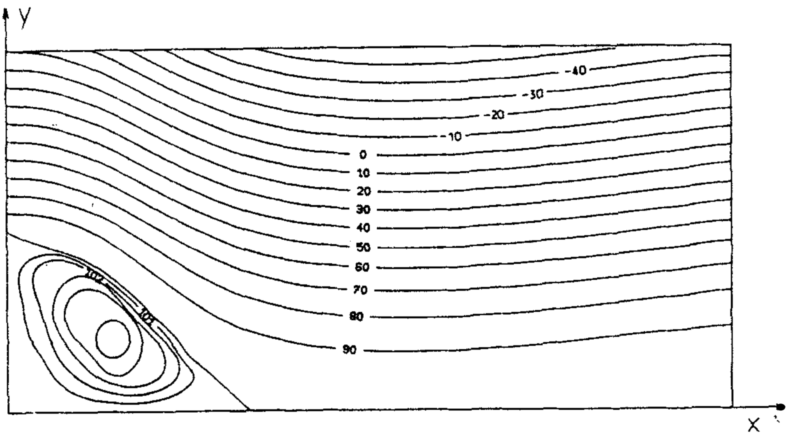

Figure 23. Flow down a step: FTUS1 scheme, Re = 1,000,000

6.3. Discussion

Turbulent flows over a step were studied experimentally, in particular by Honji,[10] for Re = 300, and by Abbott and Kline,[9] for $Re \simeq 2 \times 10^4$. These two studies show that the dimensionless length of re-attachment is of the order of six and that it does not vary appreciably with the Reynolds' number. In the present study this occurs in laminar flow as evaluated by the FTUS1 scheme. This proves that the overall diffusion of the numerical scheme is of the same order of magnitude as the apparent diffusion of a turbulent source. However, it is quite evident that a phenomenon of strict numerical origin, which only depends on the advective scheme and mesh size, can at no time model a physical phenomenon such as turbulence. On the contrary, the flows obtained with the FTUS3 scheme show a significant variation in terms of Reynolds' number which is in fact the main characteristic of a

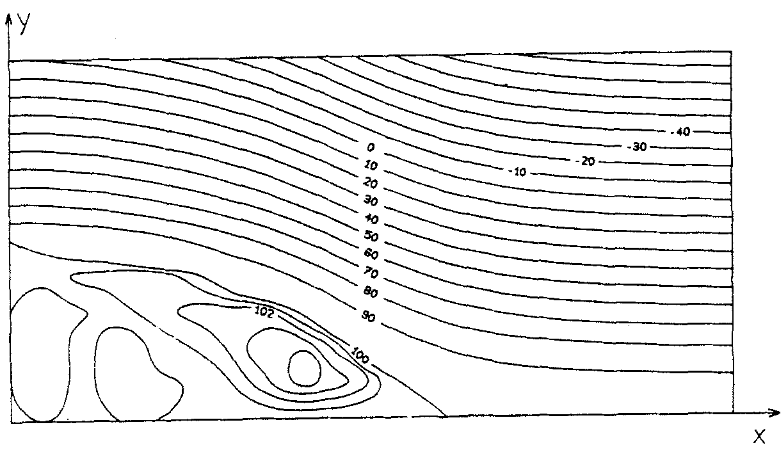

Figure 24. Flow down a step: FTUS3 scheme, Re = 1,000,000

laminar flow. Moreover, an instability area, just downstream of the step (Figure 22), is obtained for a high Reynolds' number. Now, three-dimensional flows are experimentally observed at the same place. This may confirm that there is no stable two-dimensional solution in this area.

The higher values of the dimensionless length of re-attachment point observed with the FTUS3 scheme indicate a reduced numerical diffusion, which is smaller than the turbulent diffusion since the latter converges on a value of six. A fair description of the actual phenomena should be obtained when associating a convenient model of turbulence to the computation technique advocated.

7. CONCLUSION

The FTUS1 scheme obviously induces an unacceptable numerical diffusion in transient flow conditions. It is therefore essential to develop a less diffusive scheme associated with reasonable cost. Considering the advection problem from a Lagrangian approach, some schemes are proposed which give very good results—as shown by the one-dimensional test case. Amongst them, the FTUS3 scheme (upwind differencing interpolation, third-degree polynomial) appears to be particularly interesting because of its linearity and relative programming simplicity.

By further investigations, a method of characteristics is proposed for solving the Navier–Stokes equations in primitive variables V, p. This method is of easy coding, due to the Lagrangian formalism in V^*. It has a higher accuracy when compared to the usually advocated methods, due to the two-dimensional interpolation and computation of the curvilinear characteristic.

In steady state, the distortions induced by the numerical diffusion are more difficult to evaluate. In actual fact, they occur in laminar solutions only with high Reynolds' number, i.e. when the actual physical flow is turbulent, thus preventing any direct comparison with the experimental data. However, a model test of flow behind sudden enlargement clearly shows that in some cases the numerical diffusion induced by the FTUS1 scheme has the same order of magnitude as the effective turbulent diffusion and it therefore appears unnecessary to associate to this scheme a sophisticated simulation of turbulence. On the contrary, the numerical diffusion of the FTUS3 scheme remains significantly smaller, thus giving full interest to the modelling of turbulence.

This is why the FTUS3 scheme appears to us quite suitable for the computation of steady and transient turbulent flows in industrial applications.

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